Multivariate risks and depth-trimmed regions

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Abstract

We describe a general framework for measuring risks, where the risk measure takes values in an abstract cone. It is shown that this approach naturally includes the classical risk measures and set-valued risk measures and yields a natural definition of vector-valued risk measures. Several main constructions of risk measures are described in this axiomatic framework.

It is shown that the concept of depth-trimmed (or central) regions from the multivariate statistics is closely related to the definition of risk measures. In particular, the halfspace trimming corresponds to the Value-at-Risk, while the zonoid trimming yields the expected shortfall. In the abstract framework, it is shown how to establish a both-ways correspondence between risk measures and depth-trimmed regions. It is also demonstrated how the lattice structure of the space of risk values influences this relationship.

AMS Classification 91B30 91B82 60D05 62H99

1 Introduction

Risk measures are widely used in financial engineering to assess the risk of investments and to optimise the capital allocation. The modern theory of coherent risk measures [2, 9] aims to derive properties of risk measures from several basic axioms: translation-invariance,

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monotonicity, homogeneity and convexity. The risk measures are mostly considered in the univariate case, i.e. it is assumed that all assets have been transferred to their monetary values. The quantile-based risk measures gain a particular importance in the form of so-called spectral risk measures that are weighted integrals of the quantile function, see [1].

When assessing risks of multivariate portfolios, the situation becomes more complicated. The quantile function is not a numerical function any more, and it is not possible to represent all portfolios as functions of a uniform random variable. The simplest approach to assess the risk of a multivariate portfolio is to aggregate the individual assets using their cash equivalents and then assess the risk of the combined univariate portfolio. Then all portfolios with identically distributed monetary equivalents would have identical risks.

Several recent papers suggest various alternative ways of measuring risks for multivariate portfolios without taking their monetary equivalents. The multivariate analogue of the Value-at-Risk discussed in [10] is based on set-valued quantiles of the multivariate cumulative distribution function. A construction of real-valued multivariate risk measures based on combining univariate risks from transformed portfolios is described in [4]. Multivariate coherent risk measures have been studied in [16] following the techniques from [9] based on the duality representations. The risk measures considered in [16] are actually set-valued and the preference order corresponds to the ordering of sets by inclusion. It is interesting to note that this order has the same meaning for risk, but formally is the exactly opposite to the ordering of univariate risks from [2]. Set-valued risk measures have been also studied in [12].

Because of this reason and in order to unify several existing definitions we decided to consider risk measures as maps that have values in a certain partially ordered cone, which may be, e.g., the real line or the Euclidean space or the family of convex sets in the Euclidean space. We single out the basic properties of so defined risk measures and then describe the main technical constructions that make it possible to produce new risk measures from the existing ones while respecting their properties, e.g. the homogeneity or coherence. It is not always assumed that the risk measures are coherent. Note that risk measures with values in a partially ordered cone have been considered in [15], where however it was assumed that this cone is embeddable into a linear space. This is not the case for set-valued risk measures which are also covered by the current work. These set-valued measures can be used to produce vector-valued or real-valued risk measures for multivariate portfolios.

In comparison with the studies of multivariate risk measures, the multivariate statistical theory has an impressive toolbox suitable to handle random vectors. We show that the multivariate setting for the risk measures has a number of common features with the concept of central (or depth-trimmed) regions well known in multivariate statistics [24, 25]. They associate a random vector with a set formed by the points in space located near to the "central value" of this random vector. The risk measure is generated by considering all translations of a random vector that bring its central region to the positive (acceptable) part of the space. In other words, the risks is determined by the relative location of the central region comparing to the acceptable or completely non-acceptable risks. Note that in the multivariate setting the sets of acceptable and non-acceptable risk values are no longer complementary, as they

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are in the univariate setting. Estimation methods for depth trimmed regions then may be utilised to come up with estimators for multivariate risk measures. Despite the fact that the definition of central regions (and indeed the name also) treats all directions in the same way, it is possible to establish a two-way link between depth-trimmed regions and risk measures.

The paper is organised as follows. Section 2 introduces the main concept of a risk measure with values in an abstract cone. As special cases one obtains the classical risk measures [2], set-valued risk measures of [16] and vector-valued risk measures. A crucial concept here is the function that assigns risks to deterministic outcomes and controls changes of the risk if a deterministic amount is being added to a portfolio. The partial order relation on the space of risks makes it possible to consider it as a lattice.

The acceptance cone constitutes a subset of acceptable values for the risk measure, while the acceptance set is the family of random vectors whose risks belong to the acceptance cone. Section 3 discusses the main properties of the acceptance set and the acceptance cone. We single out conditions that make it possible to retrieve the risk measure from the acceptance set it generates. This self-consistency condition can be traced to some facts from the morphological theory of lattices [13].

Section 4 describes several ways to construct new risk measures: minimisation, re-centring, homogenisation, worst conditioning and transformations of risks. In particular, the worst conditioning is a generic construction that yields the expected shortfall if applied to the expectation. It is shown that by transforming risks it is possible to produce vector-valued risk measures from set-valued risk measures. This construction can be applied, for instance, to the set-valued risk measures from [16].

The definition of depth-trimmed regions and their essential properties in view of relationships to risk measures are given in Section 5. In particular, the well-known halfspace trimmed regions [20, 23] correspond to the Value-at-Risk and the zonoid trimming [22] produces the expected shortfall. This analogy goes much further and leads to a systematic construction of a risk measure from a family of depth-trimmed regions in Section 6. The main idea here is to map the depth-trimmed region of a random vector into the risk space using the function that assigns risks to deterministic outcomes and then consider all translations of the image (of the depth-trimmed region) that place it inside the acceptance cone. Examples of basic risk measures obtained this way are described in Section 7. It is shown in Section 8 that the correspondence between risk measures and depth-trimmed regions goes both ways, i.e. it is possible to construct a family of depth-trimmed regions from a risk measure, so that, under some conditions, the initial risk measure is recoverable from the obtained family of depth-trimmed regions.

Finally, Section 9 deals with dual representation of coherent risk measures and depth-trimmed regions using families of measures, in a way similar to the well-known approach [9] for real-valued coherent risk measures. In particular we show that all coherent vector-valued risk measures are marginalised, i.e. can be represented as the vector of risk measures for the marginals. This fact confirms the idea that set-valued risk measures are a natural tool for multivariate portfolios if one is interested in non-trivial coherent risk measures.

2   Risk measures in abstract cones

A risky portfolio is modelled as an essentially bounded random vector X that represents a financial gain. Let L_d^∞ denote the set of all essentially bounded d-dimensional *random vectors* on the probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. In order to combine several definitions of risk measures, it is sensible to regard them as functionals on L_d^∞ with values in a partially ordered *convex cone* $\mathbb{C}$.

Definition 2.1 (Semigroup and convex cone). *An abelian topological semigroup is a topological space $\mathbb{C}$ equipped with a commutative and associative continuous binary operation $\oplus$. It is assumed that $\mathbb{C}$ possesses the* neutral element $\mathbf{e}$ *satisfying $x \oplus \mathbf{e} = x$ for all $x \in \mathbb{C}$. The semigroup $\mathbb{C}$ is a* convex cone *if it is also equipped with a continuous operation $(x, t) \mapsto t \odot x$ of multiplication by positive scalars $t > 0$ for $x \in \mathbb{C}$ so that $1 \odot x = x$ for all $x \in \mathbb{C}$, $t \odot \mathbf{e} = \mathbf{e}$ for all $t > 0$, and the following conditions are satisfied*

$$t \odot (x \oplus y) = t \odot x \oplus t \odot y, \quad t > 0, \ x, y \in \mathbb{C},$$
$$t \odot (s \odot x) = (ts) \odot x, \quad t, s > 0, \ x \in \mathbb{C}.$$

Assume throughout that $\mathbb{C}$ is endowed with a *partial order* $\preceq$ that is compatible with the (commutative) addition operation and multiplication by scalars, i.e. $x \preceq y$ implies that $x \oplus z \preceq y \oplus z$ for all z and $t \odot x \preceq t \odot y$ for all $t > 0$. Furthermore, assume that $\mathbb{C}$ with the order $\preceq$ is a *complete lattice*, i.e. every set has supremum and infimum, which are denoted by $\vee$ and $\wedge$ respectively. Since this partial order may differ from the conventional order for real numbers, we retain the notation supremum and infimum (also min and max) for the conventional order on the real line, while $\vee$ and $\wedge$ denote the supremum and infimum in $\mathbb{C}$. The *top element* of $\mathbb{C}$ is denoted by $\mathbb{T}$. It is assumed that the top element is absorbing, i.e. $\mathbb{T} \oplus a = \mathbb{T}$ for all $a \in \mathbb{C}$.

Note that the cone $\mathbb{C}$ is not necessarily embeddable in a linear space, since the addition operation does not necessarily obey the cancellation law and the second distributivity law $t \odot x \oplus s \odot x = (t + s) \odot x$ is not imposed, see [8] for a discussion of algebraic properties of convex cones. Accordingly, it is not possible to view $\mathbb{C}$ as a partially ordered linear space. This situation is typical if $\mathbb{C}$ is the family of convex sets in the Euclidean space $\mathbb{R}^d$ and the additive operation is the *closed Minkowski addition*, i.e. the sum of $A \oplus B$ of two sets is the topological closure of $\{x + y : \ x \in A, \ y \in B\}$. Note that the Minkowski sum of two non-compact closed sets is not necessarily closed. The multiplication by positive numbers is given by $tA = \{tx : \ x \in A\}$, i.e. the usual dilation of A by $t > 0$ and we simply write $x + A$ instead of $\{x\} \oplus A$.

We retain the usual $+$ and multiplication signs for operations with real numbers and vectors in $\mathbb{R}^d$. For convenience, letters x, y, z with or without subscripts stand for points in $\mathbb{R}^d$, letters t, s represent real numbers, letters a, b denote elements of $\mathbb{C}$, letters X, Y are used for random variables or random vectors, and A, B, F, K are subsets of $\mathbb{R}^d$.

A *proper Euclidean convex cone* K is a strict subset of $\mathbb{R}^d$ such that $\mathbb{R}_+^d \subseteq K$, K does not contain any line, and $x + y \in K$, $tx \in K$ for all $x, y \in K$ and $t > 0$. In the univariate case $(d = 1)$ the only possibility is $K = [0, \infty)$.

Definition 2.2 (Order in $\mathbb{R}^d$). *Let K be a proper Euclidean convex cone. For $x, y \in \mathbb{R}^d$, we denote $x \leq_K y$ if and only if $y - x \in K$.*

From the economical viewpoint, this ordering would correspond, e.g. to exchanges of various currencies, cf. [16, 17].

A risk measure is a functional on L_d^∞ with values in $\mathbb{C}$. As the first step of its proper definition, one should specify how this functional acts on degenerate random variables, i.e. on the space $\mathbb{R}^d$, which is naturally embedded in L_d^∞. This action is defined by a function $f : \mathbb{R}^d \mapsto \mathbb{C}$, which is interpreted as the risk associated with the degenerate random variable $X = x$ a.s. Assume that $f(0) = \mathbf{e}$, f is linear, i.e.

$$f(x) \oplus f(y) = f(x + y) \tag{2.1}$$

for all $x, y \in \mathbb{R}^d$, and non-decreasing, i.e. $f(y) \preceq f(x)$ if $y \leq_K x$. The mapping f is a linear positive map between partially ordered linear spaces: $\mathbb{R}^d$ with the $\leq_K$ order and the space $\mathbb{F} = \{f(y) : y \in \mathbb{R}^d\}$ with the order inherited from $\mathbb{C}$. Condition (2.1) implies that $f(x) \neq \mathbb{T}$ for all x. Indeed, if $f(x) = \mathbb{T}$, then $f(x + y) = \mathbb{T}$ for all y, so that f identically equals $\mathbb{T}$ contrary to the fact that $f(0) = \mathbf{e}$.

The following definition specifies the desirable properties of risk measures.

Definition 2.3 (Risk measure). *A functional $\varrho : L_d^\infty \mapsto \mathbb{C}$ is called a* risk measure *associated with f if $\varrho(X) = f(x)$ in case $X = x$ a.s. and the following conditions hold*

R1 $f(y) \oplus \varrho(X) = \varrho(X + y)$ *for all $y \in \mathbb{R}^d$;*

R2 $\varrho(Y) \preceq \varrho(X)$ *whenever $Y \leq_K X$ a.s.;*

is called a homogeneous *risk measure if also*

R3 $\varrho(tX) = t \odot \varrho(X)$ *for all $t > 0$ and $X \in L_d^\infty$;*

and a coherent *risk measure if additionally*

R4 $\varrho(X) \oplus \varrho(Y) \preceq \varrho(X + Y)$ *for all $X, Y \in L_d^\infty$.*

Condition **R2** means that ϱ is a lattice morphism between L_d^∞ with the partial order generated by $\leq_K$ and $\mathbb{C}$. It is also possible to consider not necessarily homogeneous risk measures that satisfy the assumption

$$t \odot \varrho(X) \oplus (1 - t) \odot \varrho(Y) \preceq \varrho(tX + (1 - t)Y) \tag{2.2}$$

for all $t \in [0, 1]$, which are traditionally called *convex* [11] (despite the fact that the inequality in our setting actually means that ϱ is concave).

Note that the multiplication by numbers in $\mathbb{C}$ is not needed if **R3** is not considered. In this case one can only require that $\mathbb{C}$ is a partially ordered abelian semigroup. Furthermore, Definition 2.3 can be formulated for any partially ordered cone $\mathbb{C}$ (not necessarily a complete lattice) and any partial ordering on $\mathbb{R}^d$.

Since $\mathbf{e} \neq \mathbb{T}$, the condition $f(0) = \mathbf{e}$ together with **R2** imply that $\varrho(X)$ never takes the value $\mathbb{T}$. This corresponds to the requirement that conventional risk measures do not take the value $-\infty$, see [9]. Indeed, if $\varrho(X) = \mathbb{T}$, then $f(a) = \varrho(a) = \mathbb{T}$ for a being an upper bound for X.

The use of function f in Definition 2.3 is twofold. It determines risks of deterministic portfolios and also controls how the risk of X changes if a deterministic quantity is added to the portfolio X. The second task can be also delegated to another function $g : \mathbb{R}^d \mapsto \mathbb{C}$, so that **R1** becomes $g(y) \oplus \varrho(X) = \varrho(X + y)$ and $g(y) \oplus g(-y) = \mathbf{e}$ for all $y \in \mathbb{R}^d$. It is easy to show that f and g coincide if and only if $f(0) = \mathbf{e}$.

Example 2.4 (Set-valued risk measures). Consider the family of closed convex sets in $\mathbb{R}^d$ partially ordered by inclusion with the addition defined as the closed Minkowski sum and the conventional dilation by positive numbers. Define $f(x) = \{y \in \mathbb{R}^d : -x \leq_K y\} = -x + K$, where K is a proper Euclidean cone from Definition 2.2. In particular, the fact that $\varrho(X) \supset K$ means that X has a negative risk. In this case Definition 2.3 turns into [16, Def. 2.1]. Since $f(0) = K$ has to be the neutral element, the relevant cone $\mathbb{C}$ should consist of all closed convex sets $F \subseteq \mathbb{R}^d$ such that the closed Minkowski sum $F \oplus K$ coincides with F. This important family of sets will be denoted by $\mathbb{C}_K$.

Let us show that $\mathbb{C}_K$ is a complete lattice. Consider any family of sets $\{A_i : i \in I\} \subseteq \mathbb{C}_K$. Then $F = \bigvee_{i \in I} A_i$ is the smallest convex set that contains all the A_is, i.e. F is the closure of the convex hull of the union of these sets. Since F is closed convex and $F = \bigvee(A_i \oplus K) = K \oplus F$, we have $F \in \mathbb{C}_K$. Furthermore, $M = \bigwedge_{i \in I} A_i$ is given by $M = \bigcap_{i \in I} A_i$. The set M is closed convex and also belongs to $\mathbb{C}_K$, since

$$M = \bigcap_{i \in I} A_i = \bigcap_{i \in I}(A_i \oplus K) \supseteq K \oplus \bigcap_{i \in I} A_i \supseteq M \,,$$

because K contains the origin.

Example 2.5 (Univariate risk measures). The classical definition of real-valued coherent risk measures from Artzner *et al.* [2] can be recovered from the setting of Example 2.4 for $d = 1$ and $\varrho(X) = [\rho(X), \infty)$, where $\rho(X)$ is the risk measure of X as in [2]. An alternative approach is to let $\mathbb{C}$ be the extended real line $\overline{\mathbb{R}} = [-\infty, \infty]$ with the *reversed* order and conventional addition and multiplication operations. In this case $f(x) = -x$. We will briefly recall three univariate risk measures: the value at risk, which is the most widely used risk measure, and two coherent risk measures, the expected shortfall and the expected minimum.

The *value at risk* is defined as the amount of extra capital that a firm needs in order to reduce the probability of going bankrupt to a fixed threshold α. It is the opposite of the α-quantile of a random variable X, i.e.

$$\mathrm{V@R}_\alpha(X) = -\inf\{x : \mathbf{P}\{X \leq x\} > \alpha\} = -F_X^{-1}(\alpha) \,,$$

where F_X is the cumulative distribution function of X. It can be shown that the value at risk is a homogeneous risk measure, but not a coherent one. It satisfies properties **R1**, **R2** and **R3**, but not necessarily **R4**.

The *expected shortfall* is a coherent risk measure defined as

$$\mathrm{ES}_\alpha(X) = -\frac{1}{\alpha} \int_0^\alpha F_X^{-1}(t)\mathrm{d}t\,,$$

where $\alpha \in (0,1]$.

The *expected minimum* is another coherent risk measure defined as

$$\mathrm{EM}_{1/n}(X) = -\mathbf{E}\min\{X_1, X_2, \ldots, X_n\},$$

where $X_1, X_2, \ldots, X_n$ are independent copies of X. The expected minimum belongs to the family of weighted V@Rs and is called Alpha-V@R in [7].

In the following we often consider the Euclidean space $\mathbb{R}^d$ extended by adding to it the top and bottom elements at the infinity, so that the space then becomes a complete lattice. In order to simplify notation we retain notation $\mathbb{R}$ and $\mathbb{R}^d$ for such extended spaces.

Example 2.6 (Marginalised multivariate vector-valued risk measures). Let $\mathbb{C}$ be $\mathbb{R}^d$ with the usual addition, multiplication by positive numbers and the reversed coordinatewise order, i.e. $a \preceq b$ if $b \leq_K a$ with $K = \mathbb{R}_+^d$. Given a d-dimensional random vector $X = (X_1, \ldots, X_d)$, any of the aforementioned univariate risk measures ρ yields a risk measure $\varrho(X) = (\rho(X_1), \ldots, \rho(X_d))$ with values in $\mathbb{R}^d$. In this case $f(x) = -x$.

3 Acceptance cones and acceptance sets

The concept of an acceptance set is the dual one to the risk measure, see [2, 11, 16]. The main idea is that a portfolio X is acceptable if $\varrho(X)$ belongs to a certain subcone $\mathbb{A} \subset \mathbb{C}$ called the *acceptance cone*. The classical setting (see Example 2.5) corresponds to $\mathbb{C} = \mathbb{R}$ with the reversed order and $\mathbb{A} = (-\infty, 0]$. Every acceptance cone $\mathbb{A}$ is upper with respect to $\preceq$, i.e. if $a \preceq b$ and $a \in \mathbb{A}$, then $b \in \mathbb{A}$. We also assume that

$$\{a \in \mathbb{C}: \ \mathbf{e} \preceq a\} = \mathbb{A}\,, \tag{3.1}$$

i.e. a deterministic portfolio x is acceptable if and only if $0 \leq_K x$.

Given the risk measure ϱ, the set $\mathcal{A} \subset L_d^\infty$ of acceptable portfolios (called the *acceptance set*) is given by

$$\mathcal{A} = \{X \in L_d^\infty: \ \varrho(X) \in \mathbb{A}\} = \{X \in L_d^\infty: \ \mathbf{e} \preceq \varrho(X)\}\,.$$

If ϱ is coherent, then $\mathcal{A}$ is a cone in L_d^∞. It follows from **R1** that

$$\{y: \ X - y \in \mathcal{A}\} = \{y: \ \varrho(X - y) \in \mathbb{A}\} = \{y: \ \varrho(X) \oplus f(-y) \in \mathbb{A}\}\,.$$

The f-image of the set in the right-hand side is

$$\begin{aligned} \varrho_{\mathcal{A}}(X) &= \{f(y): \ y \in \mathbb{R}^d, \ \varrho(X) \oplus f(-y) \in \mathbb{A}\} \\ &= \{a \in \mathbb{F}: \ \varrho(X) \in \mathbb{A} \oplus a\} \\ &= \{a \in \mathbb{F}: \ a \preceq \varrho(X)\}\,. \end{aligned}$$

Indeed, since the family $\mathbb{F}$ of values of f is a linear space, $\mathbb{A} \oplus a = \{b \oplus a : b \in \mathbb{A}\}$ coincides with the set $\{b \in \mathbb{C} : a \preceq b\}$ for any $a \in \mathbb{F}$.

Note that $\varrho_{\mathcal{A}}(X)$ is not necessarily an element of $\mathbb{C}$, since it may consist of several elements of $\mathbb{C}$. For instance, in Example 2.5 (with $\mathbb{C} = \mathbb{R}$), $\varrho_{\mathcal{A}}(X)$ is the set $[\rho(X), \infty)$, while the risk of X is a real number. In this case, one can retrieve the risk of X by taking the infimum of all members of $\varrho_{\mathcal{A}}(X)$. This minimum corresponds to the $\vee$-operation in $\mathbb{R}$ with the reversed order. The following easy observation generalises the well-known relationship between risk measures and acceptance sets [2, 9].

Proposition 3.1. *If $\mathbb{F}$ is sup-generating (see [13, p. 28]), i.e.*

$$b = \bigvee \{a \in \mathbb{F} : a \preceq b\} \quad \text{for all } b \in \mathbb{C}, \tag{3.2}$$

then

$$\varrho(X) = \bigvee \varrho_{\mathcal{A}}(X).$$

In the multivariate case one often needs the concept of the rejection cone $\mathbb{A}^{\mathrm{r}} = \{a \in \mathbb{C} : a \preceq \mathbf{e}\}$ and the rejection set

$$\mathcal{A}^{\mathrm{r}} = \{X \in L_d^\infty : \varrho(X) \preceq \mathbf{e}\}.$$

While $\mathbb{A}^{\mathrm{r}}$ is a subcone of $\mathbb{C}$, the set $\mathcal{A}^{\mathrm{r}}$ is not necessarily convex even if ϱ is coherent. Indeed, if $X, Y \in \mathcal{A}^{\mathrm{r}}$, then $\varrho(X) + \varrho(Y) \preceq \mathbf{e}$, while **R4** no longer suffices to deduce that $\varrho(X+Y) \preceq \mathbf{e}$.

Example 3.2 (Set-valued risk measures). Let $\mathbb{C}_K$ be the cone of convex closed sets described in Example 2.4 and $f(x) = -x + K$, so that $\mathbb{F} = \{y + K : y \in \mathbb{R}^d\}$. If $\mathbb{A} = \{A \in \mathbb{C}_K : K \subseteq A\}$, then $\mathbb{F}$ is sup-generating, since for any $F \in \mathbb{C}_K$ we have

$$\bigvee \{a \in \mathbb{F} : a \preceq b\} = \bigcup \{y + K : y \in \mathbb{R}^d, \ (y + K) \subseteq F\} = F.$$

As in [16, Sec. 2.5], it is possible to choose another acceptance cone $\mathbb{A}'$ which is richer than the cone $\mathbb{A}$ defined above. Furthermore, the sup-generating property (3.2) corresponds to the self-consistency property from [16, Property 3.4].

Example 3.3 (Alternative construction of set-valued risk measures). There is also an alternative way to introduce set-valued risk measures. Let $\mathbb{C}_K^{\mathrm{r}}$ be the family of complements to the interiors of sets from $\mathbb{C}_K$, with the addition operation induced by one from $\mathbb{C}_K$, i.e. $F_1 \oplus F_2$ is the complement to the Minkowski sum of the complements to F_1 and F_2. The neutral element $\mathbf{e} = K^{\mathrm{r}}$ is then the complement to the interior of K. If $\mathbb{C}_K^{\mathrm{r}}$ is equipped with the inclusion order, then the same arguments as in Example 2.4 confirm that $\mathbb{C}_K^{\mathrm{r}}$ is a complete lattice.

If $f(x) = x + K^{\mathrm{r}}$, $x \in \mathbb{R}^d$, then the corresponding family $\mathbb{F}$ is inf-generating (see [13, p. 28]), i.e.

$$b = \bigwedge \{a \in \mathbb{F} : b \preceq a\} \quad \text{for all } b \in \mathbb{C}.$$

In this case

$$\varrho(X) = \bigwedge \varrho_{\mathcal{A}^{\mathrm{r}}}(X), \tag{3.3}$$

where $\varrho_{\mathcal{A}^{\mathrm{r}}}(X)$ is the f-image of all $y \in \mathbb{R}^d$ such that $X - y \in \mathcal{A}^{\mathrm{r}}$.

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Example 3.4 (Vector-valued risk measures from scalar portfolios). Consider a risk measure ϱ defined on L_1^∞ with values in $\mathbb{C} = \mathbb{R}^2$ with the usual summation and multiplication by scalars and the reversed coordinatewise ordering, i.e. the reversed ordering generated by $K = \mathbb{R}_+^2$. Such a risk measure may be defined as a vector composed of several univariate risk measures from Example 2.5. In this case $f(x) = (-x, -x)$, so that $\mathbb{F}$ is the diagonal in $\mathbb{R}^2$, which is clearly not sup-generating.

This example explains, by the way, why in the framework of [16] only risk measures that do not increase the dimension of the portfolios have been studied.

4   Constructions of risk measures

4.1   Minimisation

Consider a family ϱ_i, $i \in I$, of risk measures on the same cone $\mathbb{C}$, all associated with the same function f. Then $\varrho = \bigwedge_{i \in I} \varrho_i$ is also a risk measure associated with f. If all ϱ_i are coherent (resp. homogeneous or convex) the resulting risk measure is coherent (resp. homogeneous or convex). The acceptance set associated with ϱ is the intersection of the acceptance sets of the risk measures ϱ_i, $i \in I$.

Example 4.1 (Minimisation of univariate risk measures). While it is not interesting to take minimum of, say, the expected shortfalls at different levels, it is possible to combine members from different families of univariate risk measures. For instance, if $n \geq 1$ and $\alpha \in (0, 1]$, then $\max\{\mathrm{EM}_{1/n}(X), \mathrm{ES}_\alpha(X)\}$ is a coherent risk measure associated with $f(x) = -x$. Note that the maximum of two risk measures correspond to the minimum in $\mathbb{C} = \mathbb{R}$ with the reversed order.

4.2   Re-centring

All random vectors from L_d^∞ can be naturally centred by subtracting their expected values. This makes it possible to define a risk measure on centred random vectors and then use **R1** to extend it onto the whole L_d^∞. If ϱ is defined on the family of essentially bounded random vectors with mean zero, then the re-centred risk measure is given by

$$\varrho_\mathrm{o}(X) = \varrho(X - \mathbf{E}X) \oplus f(\mathbf{E}X), \quad X \in L_d^\infty.$$

If $\mathbb{C}$ is $\mathbb{R}^d$ or a family of subsets of $\mathbb{R}^d$, we rely on the canonical choice of the translation by setting $\varrho_\mathrm{o}(X) = \varrho(X - \mathbf{E}X) - \mathbf{E}X$.

It should be noted that **R2** does not hold automatically for re-centred risk measures and has to be checked every time the re-centring is applied.

4.3 Homogenisation

If ϱ satisfies **R1** and **R2**, it is possible to construct a homogeneous risk measure from it by setting

$$\varrho_{\mathrm{h}}(X) = \bigwedge_{t>0} \frac{1}{t} \odot \varrho(tX). \tag{4.1}$$

Note that the infimum operation $\bigwedge$ in $\mathbb{C}$ makes sense, since $\mathbb{C}$ is a complete lattice. It is easy to see that ϱ_{h} satisfies **R3**. Furthermore, it satisfies **R2** and **R1** if f is homogeneous. The latter is clearly the case if $f(x) = -x + K$, $x \in \mathbb{R}^d$, for a proper cone K, see Example 2.4.

A similar construction produces a translation-invariant risk measure from a general one by

$$\varrho_{\mathrm{t}}(X) = \bigwedge_{z\in\mathbb{R}^d} \Big(\varrho(X+z) \oplus f(-z) \Big). \tag{4.2}$$

Both (4.1) and (4.2) applied together to a function ϱ that satisfies **R2** and **R4** yield a coherent risk measure.

Example 4.2. If $\mathbb{C}$ is the real line with the reversed order and (4.1) results in a non-trivial function, then $\varrho(tX) \to 0$ as $t \to 0$. Similarly, a non-trivial result of (4.2) yields that $\varrho(X+z) \to -\infty$ as $z \to \infty$. For instance, these constructions produce trivial results if applied to the risk measure $\mathbf{E}(k-X)_+$ studied in [14].

4.4 Worst conditioning

A single risk measure ϱ can be used to produce a family of risk measures by taking the infimum of the risks associated to the random vectors obtained after certain rearrangements of the underlying probability measure. For each $\alpha \in (0,1]$ define

$$\varrho_\alpha(X) = \bigwedge_{\phi\in\Phi_\alpha} \varrho(X_\phi),$$

where $X_\phi = X \circ \phi$ and Φ_α is the family of measurable mappings $\phi : \Omega \mapsto \Omega$ such that $\mathbf{P}(\phi^{-1}(A)) \leq \alpha^{-1}\mathbf{P}(A)$ for all $A \in \mathfrak{F}$. If $X \in L_d^\infty$, then $X_\phi \in L_d^\infty$ for any $\alpha \in (0,1]$ and $\phi \in \Phi_\alpha$. It is possible to define the worst conditioning alternatively as

$$\varrho_\alpha(X) = \bigwedge_{Y\in\mathcal{P}_\alpha(X)} \varrho(Y),$$

where $\mathcal{P}_\alpha(X)$ is the family of all random vectors Y such that $\mathbf{P}\{Y \in B\} \leq \alpha^{-1}\mathbf{P}\{X \in B\}$ for all Borel $B \subset \mathbb{R}^d$.

It is easy to show that ϱ_α preserves any property that ϱ satisfies from **R1**–**R4**. For instance, if $Y \leq_K X$ a.s., then $Y_\phi \leq_K X_\phi$ a.s. for any $\phi \in \Phi_\alpha$, so that

$$\varrho_\alpha(Y) = \bigwedge_{\phi\in\Phi_\alpha} \varrho(Y_\phi) \preceq \bigwedge_{\phi\in\Phi_\alpha} \varrho(X_\phi) = \varrho_\alpha(X)$$

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whenever ϱ satisfies **R2**. If $X, Y \in L_d^\infty$ and ϱ satisfies **R4**, then

$$\varrho_\alpha(X + Y) = \bigwedge_{\phi \in \Phi_\alpha} \varrho\big((X+Y)_\phi\big) = \bigwedge_{\phi \in \Phi_\alpha} \varrho(X_\phi + Y_\phi)$$

$$\succeq \bigwedge_{\phi \in \Phi_\alpha} \big(\varrho(X_\phi) \oplus \varrho(Y_\phi)\big) \succeq \bigwedge_{\phi \in \Phi_\alpha} \varrho(X_\phi) \oplus \bigwedge_{\phi \in \Phi_\alpha} \varrho(Y_\phi) = \varrho_\alpha(X) \oplus \varrho_\alpha(Y) \,.$$

Consider now the setting of univariate risk measures from Example 2.5, i.e. X is a random variable from L_1^∞ and $\mathbb{C}$ is the real line with the reversed order. The simplest coherent risk measure is the opposite of the expectation of a random variable. In fact, this risk measure appears from the expected shortfall when $\alpha = 1$, i.e. $\mathrm{ES}_1(X) = -\mathbf{E}X$. The worst conditioning applied to the opposite of the expectation yields

$$(-\mathbf{E})_\alpha(X) = \sup_{\phi \in \Phi_\alpha} \{-\mathbf{E}(X_\phi)\} = - \inf_{\phi \in \Phi_\alpha} \mathbf{E}(X_\phi) = - \inf_{\phi \in \Phi_\alpha} \int X(\phi(\omega))\mathbf{P}(\mathrm{d}\omega)$$

$$= - \inf_{\phi \in \Phi_\alpha} \int X(\omega)\mathbf{P}\phi^{-1}(\mathrm{d}\omega) = - \inf_{\phi \in \Phi_\alpha} \mathbf{E}_{\mathbf{P}\phi^{-1}}X \,,$$

where $\mathbf{E}_{\mathbf{P}\phi^{-1}}$ denotes the expectation with respect to the probability measure $\mathbf{P}\phi^{-1}$.

In general, $-\inf_{\phi \in \Phi_\alpha} \mathbf{E}_{\mathbf{P}\phi^{-1}}X \leq \mathrm{ES}_\alpha(X)$ with the equality if $(\Omega, \mathfrak{F}, \mathbf{P})$ is non-atomic. Without loss of generality assume that $\Omega = [0, 1]$, $\mathbf{P}$ is the Lebesgue measure restricted to $[0, 1]$ and X is increasing mapping from $[0, 1]$ into $\mathbb{R}$, which implies that $X(\omega) = F_X^{-1}(\omega)$ for all $\omega \in [0, 1]$, where F_X is the cumulative distribution function of X. The infimum of $\mathbf{E}_{\mathbf{P}\phi^{-1}}X$ over all $\phi \in \Phi_\alpha$ is achieved when $X \circ \phi$ takes the smaller possible values with the highest possible probabilities, and thus it is attained at $\phi'(\omega) = \alpha\omega$. We conclude

$$(-\mathbf{E})_\alpha(X) = - \int X(\alpha\omega)\mathrm{d}\omega = -\frac{1}{\alpha} \int_0^\alpha F_X^{-1}(t)\mathrm{d}t = \mathrm{ES}_\alpha(X) \,,$$

i.e. the expected shortfall appears by applying the worst conditioning construction to the opposite of the expectation.

Example 4.3 (Worst conditioning of the expected shortfall). Let us now apply the worst conditioning to the expected shortfall at level β,

$$\big(\mathrm{ES}_\beta\big)_\alpha(X) = \sup_{\phi_1 \in \Phi_\alpha} \mathrm{ES}_\beta(X_{\phi_1}) = \sup_{\phi_1 \in \Phi_\alpha} \Big(- \inf_{\phi_2 \in \Phi_\beta} \mathbf{E}_{\mathbf{P}\phi_1^{-1}}X_{\phi_2} \Big)$$

$$= - \inf_{\phi_1 \in \Phi_\alpha, \, \phi_2 \in \Phi_\beta} \mathbf{E}_{\mathbf{P}\phi_1^{-1}\phi_2^{-1}}X \,.$$

Clearly $\phi_2 \circ \phi_1 \in \Phi_{\alpha\beta}$ and thus $\big(\mathrm{ES}_\beta\big)_\alpha(X) \leq \mathrm{ES}_{\alpha\beta}(X)$. If the probability space is non-atomic, all mappings from $\Phi_{\alpha\beta}$ can be written as the composition of a mapping from Φ_α and a mapping from Φ_β, so that $\big(\mathrm{ES}_\beta\big)_\alpha(X) = \mathrm{ES}_{\alpha\beta}(X)$. One can say that the expected shortfall is stable under the worst conditioning.

Example 4.4 (Worst conditioned V@R$_\alpha$). Let us finally apply the worst conditioning construction to the value at risk at level β considered on a non-atomic probability space $\Omega = [0, 1]$ with $\mathbf{P}$ being the Lebesgue measure. Without loss of generality assume that X is increasing, so that $X(\omega) = F_X^{-1}(\omega)$. The infimum below is attained at $\phi'(\omega) = \alpha\omega$ and since $X_{\phi'}$ is also increasing, we have $X_{\phi'}(\omega) = F_{X_{\phi'}}^{-1}(\omega)$. Thus

$$\left(\mathrm{V@R}_\beta\right)_\alpha(X) = -\inf_{\phi \in \Phi_\alpha} F_{X_\phi}^{-1}(\beta) = -X_{\phi'}(\beta) = -X(\alpha\beta) = -F_X^{-1}(\alpha\beta) = \mathrm{V@R}_{\alpha\beta}(X)\,.$$

4.5   Transformations of risks

Risk measures with values in a cone $\mathbb{C}$ may be further transformed by mapping $\mathbb{C}$ into another cone $\mathbb{C}'$ using a map h. The aim may be to change the dimensionality (cf. [16]) or produce a vector-valued risk measure from a set-valued one. The map $h : \mathbb{C} \mapsto \mathbb{C}'$ that transforms any $\mathbb{C}$-valued risk measure ϱ, into the $\mathbb{C}'$-valued risk measure $h(\varrho(\cdot))$, will be called a *risk transformation*. If h respects the coherence property of risk measures, it will be called a *coherent map*.

Let us denote by $\preccurlyeq$ the partial order in $\mathbb{C}'$ which we assume to be compatible with the (commutative) addition operation and multiplication by scalars. The additive operation on $\mathbb{C}'$ and the multiplication by numbers will also be denoted by $\oplus$ and $\odot$ respectively. In the following result, we list the properties that a coherent map should possess. The mapping that assesses the risk of a deterministic portfolio in the new cone $\mathbb{C}'$ will be $h(f(\cdot))$. Recall that $\mathbb{F}$ denotes the family of possible values of the function f.

Proposition 4.5. *A map $h : \mathbb{C} \mapsto \mathbb{C}'$ is a risk transformation if it is*

(i) non-decreasing, i.e. $h(a) \preccurlyeq h(b)$ if $a \preceq b$;

(ii) linear on $\mathbb{F}$, i.e. $h(a \oplus b) = h(a) \oplus h(b)$ for all $b \in \mathbb{C}$ and $a \in \mathbb{F}$.

Further, h is a coherent map if h is homogeneous, i.e. $h(t \odot a) = t \odot h(a)$ for all $t > 0$ and $a \in \mathbb{C}$ and also satisfies

$$h(a) \oplus h(b) \preccurlyeq h(a \oplus b) \tag{4.3}$$

for all $a, b \in \mathbb{C}$.

Proof. Since ϱ satisfies **R1** and $f(y) \in \mathbb{F}$, we have for all $y \in \mathbb{R}^d$

$$h\big(\varrho(X + y)\big) = h\big(f(y) \oplus \varrho(X)\big) = h\big(g(y)\big) \oplus h\big(\varrho(X)\big)\,,$$

i.e. **R1** holds. Property **R2** holds because h is non-decreasing. The homogeneity of $h(\varrho(\cdot))$ is evident if h is homogeneous. If ϱ is coherent and (4.3) holds, then

$$h\big(\varrho(X)\big) \oplus h\big(\varrho(Y)\big) \preccurlyeq h\big(\varrho(X) \oplus \varrho(Y)\big) \preccurlyeq h\big(\varrho(X + Y)\big)\,.$$

$\square$ $\square$

As an immediate consequence of Proposition 4.5 we deduce that every linear non-decreasing map is coherent. Such maps between partially ordered vector spaces are called Riesz homomorphisms, see [19, Sec. 18].

Example 4.6 (Vector-valued risk measures from set-valued ones). A particularly important instance of transformations of risks arises if $\mathbb{C}_K$ is a family of convex closed subsets of $\mathbb{R}^d$ with inclusion order defined in Example 2.4 and $\mathbb{C}'$ is $\mathbb{R}^d$ with the reversed $\leq_K$-order for a proper Euclidean cone K. The reversing is needed since $y + K \subseteq z + K$ (i.e. $y + K \preceq z + K$) if and only if $z \leq_K y$.

The cone K is said to be a *Riesz cone* if $\mathbb{R}^d$ with $\leq_K$-order is a Riesz space, i.e. for every $x, y \in \mathbb{R}^d$ their supremum is well defined. It follows from [19, Th. 26.11] that each Riesz cone can be represented as $K = \{u \in \mathbb{R}^d : Au \in \mathbb{R}^d_+\}$ for a non-singular $d \times d$ matrix A with non-negative entries, i.e. $K = A^{-1}\mathbb{R}^d_+$. The matrix A can represent possible transfers between the assets so that $Y \leq_K X$ if and only if AY is coordinatewise smaller than AX.

Assume that K is a Riesz cone. Then it is easy to see that $\mathbb{C}'$ is a complete lattice. Let $h(F)$ denote the supremum of $F \subset \mathbb{R}^d$ in $\mathbb{C}'$ (i.e. the $\leq_K$-infimum of F). If ϱ is a $\mathbb{C}$-valued risk measure, then $h(\varrho(\cdot))$ is a vector-valued risk measure. Indeed, the map h is monotone and homogeneous. Since

$$h(F - y + K) = h(F - y) = h(F) - y = h(F) + h(-y + K),$$

h is linear on $\mathbb{F}$. Finally, h satisfies (4.3), since $x = h(F_1)$ and $y = h(F_2)$ imply that $F_1 + F_2 \subseteq (x_1 + x_2) + K$.

It is also possible to produce vector-valued risk measures from set-valued risk measures in the cone $\mathbb{C}^r_K$ from Example 3.3 if h is chosen to be the supremum in $\mathbb{C}'$ of the complement of $F \in \mathbb{C}^r_K$.

Example 4.7 (Linear transformations of vector-valued risk measures). Let ϱ be a risk measure on L^∞_d with values in $\mathbb{R}^d$ with the reversed $\leq_K$-order for a Riesz cone K. Note that K generates both the order on L^∞_d and on the space of values for ϱ. Then

$$\tilde{\varrho}(X) = A^{-1}\varrho(AX)$$

is a risk measure with values in $\mathbb{R}^d$ with the reversed coordinatewise order.

Example 4.8 (Scalar risk measures from vector-valued ones). Let K be a Riesz cone and $\mathbb{C} = \mathbb{R}^d$ with the reversed $\leq_K$-order. Define $\mathbb{C}' = \mathbb{R}$ with the reversed natural order. Finally, let $h(a) = \langle a, u \rangle$, where $\langle \cdot, \cdot \rangle$ is the scalar product and u belongs to the *positive dual cone* to K, i.e. $\langle u, v \rangle \geq 0$ for all $v \in K$. Clearly h is a coherent map and we obtain univariate risk measures as those of Example 2.5, but now for multivariate portfolios.

5 Depth-trimmed regions

Depth functions assign to a point its degree of centrality with respect to the distribution of a random vector, see Zuo and Serfling [24]. The higher the depth of a point is, the

more central this point is with respect to the distribution of the random vector. Depth-trimmed (or central) regions are sets of central points associated with a random vector. Given a depth function, depth-trimmed regions can be obtained as its level sets. With a d-dimensional random vector X we associate the family of *depth-trimmed regions*, i.e. sets $\mathrm{D}^\alpha(X)$, $\alpha \in (0,1]$, such that the following properties hold for all $\alpha \in [0,1]$ and all $X \in L_d^\infty$:

D1 $\mathrm{D}^\alpha(X + y) = \mathrm{D}^\alpha(X) + y$ for all $y \in \mathbb{R}^d$;

D2 $\mathrm{D}^\alpha(tX) = t\mathrm{D}^\alpha(X)$ for all $t > 0$;

D3 $\mathrm{D}^\alpha(X) \subseteq \mathrm{D}^\beta(X)$ if $\alpha \geq \beta$;

D4 $\mathrm{D}^\alpha(X)$ is connected and closed.

Note that the addition of y in **D1** and the multiplication by t in **D2** are the conventional translation and the rescaling of sets in $\mathbb{R}^d$.

These properties are similar to those discussed by Zuo and Serfling [25, Th. 3.1]. Additionally, [25] requires that the depth-trimmed regions are invariant with respect to linear transformations, i.e. $\mathrm{D}^\alpha(AX) = A\mathrm{D}^\alpha(X)$ for any nonsingular matrix A.

We will consider two additional properties of depth-trimmed regions, that, to our knowledge, have not been studied in the literature so far:

D5 if $Y \leq_K X$ a.s., then $\mathrm{D}^\alpha(X) \subseteq \mathrm{D}^\alpha(Y) \oplus K$, and $0 \in \mathrm{D}^\alpha(X) \subseteq K$ if $X = 0$ a.s.;

D6 $\mathrm{D}^\alpha(X + Y) \subseteq \mathrm{D}^\alpha(X) \oplus \mathrm{D}^\alpha(Y)$.

Observe that depth-trimmed regions are closed subsets of $\mathbb{R}^d$ and the addition operation in **D5** and **D6** is the closed Minkowski addition. Later on we will see that **D6** is closely related to the coherence property of risk measures.

Example 5.1 (Halfspace trimming). The *halfspace trimmed regions* are built as the intersection of closed halfspaces whose probability is not smaller than a given value:

$$\mathrm{HD}^\alpha(X) = \bigcap \left\{ H : \; H \text{ closed halfspace with } \mathbf{P}\{X \in H\} \geq 1 - \alpha \right\}.$$

The above definition of the halfspace trimmed regions is taken from Massé and Theodorescu [20]. Alternatively, the strict inequality in the definition of HD^α is replaced by the non-strict one, see Rousseeuw and Ruts [23]. However the definition of [20] leads to a simpler relationship between the value at risk and the univariate halfspace trimming, see Section 6.

It is well known that the halfspace trimmed regions satisfy **D1**–**D4** and they are compact and convex. The new property **D6** does not hold in general; this can be shown in the univariate case using examples for which the value at risk does not satisfy **R4**. The monotonicity property **D5** does neither hold in general.

However, it is possible to build a variant of the halfspace trimmed regions satisfying **D5**. We define the *monotone halfspace trimmed regions* as

$$\mathrm{HD}_K^\alpha(X) = \bigcap_{u \in K^*} \left\{ H_u : \; \mathbf{P}\{X \in H_u(t)\} \geq 1 - \alpha \right\}, \tag{5.1}$$

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where $H_u = \{x \in \mathbb{R}^d : \langle x, u \rangle \geq 1\}$ denotes a halfspace, and $K^* = \{u : \langle u, v \rangle \geq 0, v \in K\}$ is the positive dual cone to K. The monotone halfspace trimmed regions satisfy **D1**–**D5** and are nonempty for all $\alpha \in (0, 1]$.

Example 5.2 (Zonoid trimming). Koshevoy and Mosler [18] defined *zonoid trimmed regions* for an integrable random vector X in $\mathbb{R}^d$ as

$$\mathrm{ZD}^\alpha(X) = \left\{ \mathbf{E}[Xl(X)] : l : \mathbb{R}^d \mapsto [0, \alpha^{-1}] \text{ measurable and } \mathbf{E}l(X) = 1 \right\}, \qquad (5.2)$$

where $\alpha \in (0, 1]$. Properties **D1**–**D4** together with convexity and boundedness (and thus compactness) are already derived in [18]. The proofs of **D5** and **D6** do not involve serious technical difficulties.

Example 5.3 (Expected convex hull trimming). *Expected convex hull regions* of a random vector X at level n^{-1} for $n \geq 1$ are defined by Cascos [5] as the selection (Aumann) expectation of the convex hull of n independent copies $X_1, \ldots, X_n$ of X, see [21, Sec. 2.1] for the definition of expectation for random sets. The expected convex hull region can be given implicitly in terms of its support function as

$$h(\mathrm{CD}^{1/n}(X), u) = \mathbf{E} \max\{\langle X_1, u \rangle, \langle X_2, u \rangle, \ldots \langle X_n, u \rangle\} \quad \text{for all } u \in \mathbb{R}^d \,,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product. Note that for any $F \subset \mathbb{R}^d$ its *support function* is given by $h(F, u) = \sup\{\langle x, u \rangle : x \in F\}$ for $u \in \mathbb{R}^d$. The expected convex hull regions satisfy properties **D1**–**D6** and are compact and convex.

Example 5.4 (Integral trimming). Let $\mathcal{F}$ be a family of measurable functions from $\mathbb{R}^d$ into $\mathbb{R}$. Cascos and López-Díaz [6] defined the family of integral trimmed regions as

$$\mathrm{D}_{\mathcal{F}}^\alpha(X) = \bigcup_{Y \in \mathcal{P}_\alpha(X)} \left\{ x \in \mathbb{R}^d : \mathfrak{f}(x) \leq \mathbf{E}\,\mathfrak{f}(Y) \text{ for all } \mathfrak{f} \in \mathcal{F} \right\}$$

$$= \bigcup_{Y \in \mathcal{P}_\alpha(X)} \bigcap_{\mathfrak{f} \in \mathcal{F}} \mathfrak{f}^{-1}\big((-\infty, \mathbf{E}\,\mathfrak{f}(Y)]\big) \,,$$

where $\mathcal{P}_\alpha(X)$ is defined in Section 4.4. All families of integral trimmed regions satisfy **D3**. Other properties of the integral trimmed regions heavily depend on their generating family of functions. For instance, if for any $\mathfrak{f} \in \mathcal{F}$, $t > 0$ and $z \in \mathbb{R}^d$, the function $\mathfrak{f}_{t,z}$ defined as $\mathfrak{f}_{t,z}(x) = \mathfrak{f}(tx + z)$ belongs to $\mathcal{F}$, then the integral trimmed regions generated by $\mathcal{F}$ satisfy properties **D1** and **D2**.

If $\mathcal{F} = \{\mathfrak{f}_{t,z} : t > 0, z \in \mathbb{R}^d\}$ with continuous and $\leq_K$-decreasing function $\mathfrak{f}$, then

$$\mathrm{D}_{\mathcal{F}}^\alpha(X) = \bigcup_{Y \in \mathcal{P}_\alpha(X)} \bigcap_{\substack{t > 0 \\ z \in \mathbb{R}^d}} \left[\frac{1}{t}\Big(\mathfrak{f}^{-1}\big(\mathbf{E}\,\mathfrak{f}(tY + z)\big) - z\Big) \oplus K \right]. \qquad (5.3)$$

Hereafter we will assume that all depth-trimmed regions satisfy **D1**–**D5**.

6 Risk measures generated by depth-trimmed regions

As a motivation for the following, note that for an essentially bounded random variable X, $\alpha \in (0,1]$ and $n \geq 1$, we have

$$V@R_\alpha(X) = -\min HD^\alpha_{[0,\infty)}(X)\,,$$
$$ES_\alpha(X) = -\min ZD^\alpha(X)\,,$$
$$EM_{1/n}(X) = -\min CD^{1/n}(X)\,.$$

The following example provides another argument showing relationships between depth-trimmed regions and risk measures.

Example 6.1 (Depth-trimmed regions as set-valued risk measures). Observe that any depth-trimmed region that satisfies **D1**–**D5** can be transformed into a set-valued risk measure from Definition 2.3. Namely, $\varrho(X) = D^\alpha(X) \oplus K$ is a risk measure in the cone $\mathbb{C}_K$ of closed subsets of $\mathbb{R}^d$ with the addition operation being the closed Minkowski addition and the *reversed* inclusion order. Because of the reversed order, the function f is given by $f(x) = x + K$. However, the obtained risk measure is not coherent even if **D6** holds.

In order to construct a coherent risk measure from depth-trimmed regions, define $\varrho(X)$ to be the closure of the complement to $D^\alpha(X) \oplus K$. Then ϱ becomes a coherent risk measure in the cone $\mathbb{C}^{\mathrm{r}}_K$ from Example 3.3 if the depth-trimmed region satisfies **D1**–**D6**.

In general, a random portfolio X will be acceptable or not depending on the depth-trimmed region of level α associated with X. Since the depth-trimmed regions are subsets of the space $\mathbb{R}^d$ where X takes its values, we need to map it into the space $\mathbb{C}$ where risk measures take their values. This map is provided by the function f from Definition 2.3. Then

$$\mathbb{D}^\alpha(X) = f(D^\alpha(X) \oplus K)$$

is a subset of $\mathbb{C}$. Recall that the acceptance cone $\mathbb{A}$ is a subset of $\mathbb{C}$ that characterises the acceptable values of the risk measure, see (3.1).

Definition 6.2. *The* acceptance set at level α *associated with the depth-trimmed region* $D^\alpha(\cdot)$ *and function f is defined as*

$$\mathcal{A}_\alpha = \{X \in L^\infty_d : \mathbb{D}^\alpha(X) \subseteq \mathbb{A}\}. \tag{6.1}$$

Theorem 6.3. *The acceptance sets associated with depth-trimmed regions satisfy the following properties:*

  (i) $0 \in \mathcal{A}_\alpha$ *for all α;*

  (ii) if $\alpha \geq \beta$, then $\mathcal{A}_\beta \subseteq \mathcal{A}_\alpha$;

  (iii) if $X \in \mathcal{A}_\alpha$, then $tX \in \mathcal{A}_\alpha$ for all $t > 0$;

  (iv) if $X \in \mathcal{A}_\alpha$ and $f(x) \in \mathbb{A}$, then $x + X \in \mathcal{A}_\alpha$;

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(v) if $Y \in \mathcal{A}_\alpha$ and $Y \leq_K X$ a.s., then $X \in \mathcal{A}_\alpha$;

*(vi) if $X, Y \in \mathcal{A}_\alpha$ and **D6** holds, then $X + Y \in \mathcal{A}_\alpha$.*

Proof. *(i)* By **D5**, $\mathbb{D}^\alpha(0) = f(K) \subseteq \mathbb{A}$, i.e. $0 \in \mathcal{A}_\alpha$ for all α.
(ii) By **D3**, $\mathrm{D}^\alpha(X) \subseteq \mathrm{D}^\beta(X)$ whenever $\alpha \geq \beta$. Thus $\mathbb{D}^\alpha(X) \subseteq \mathbb{D}^\beta(X)$ and $\mathcal{A}_\beta \subseteq \mathcal{A}_\alpha$ trivially holds.
(iii) By **D2** and the homogeneity of f, we have $\mathbb{D}^\alpha(tX) = t \odot \mathbb{D}^\alpha(X)$ for all $t > 0$. Since $\mathbb{A}$ is a cone, $\mathbb{D}^\alpha(tX) \subseteq \mathbb{A}$ if and only if $\mathbb{D}^\alpha(X) \subseteq \mathbb{A}$.
(iv) Let $f(x) \in \mathbb{A}$. By **D1**, we have $\mathbb{D}^\alpha(X + x) = f(\mathrm{D}^\alpha(X) + x)$ and by (2.1), we have $\mathbb{D}^\alpha(X + x) = \mathbb{D}^\alpha(X) \oplus f(x) \subseteq \mathbb{A}$ because $\mathbb{A}$ is a (convex) cone. By (6.1), $X + x \in \mathcal{A}_\alpha$.
(v) Note that $f(\mathrm{D}^\alpha(Y) \oplus K) \subseteq \mathbb{A}$. By **D5**, $\mathrm{D}^\alpha(X) \oplus K \subseteq \mathrm{D}^\alpha(Y) \oplus K$ and thus $f(\mathrm{D}^\alpha(X) \oplus K) \subseteq \mathbb{A}$.
(vi) By (2.1) and **D6**,

$$\mathbb{D}^\alpha(X + Y) \subseteq f(\mathrm{D}^\alpha(X) \oplus \mathrm{D}^\alpha(Y) \oplus K) = \mathbb{D}^\alpha(X) \oplus \mathbb{D}^\alpha(Y) \subseteq \mathbb{A}.$$

Finally, the fact that $\mathbb{A}$ is a convex cone yields that $X + Y \in \mathcal{A}_\alpha$. □ □

Similarly to the construction used in Section 3, we measure the risk of a portfolio X in terms of the collection of deterministic portfolios x that *cancel* the risk induced by X and make $X + x$ acceptable.

Definition 6.4. *The risk measure induced by a family of depth-trimmed regions D^α at level α is given by*

$$s_\alpha(X) = \bigvee \{f(y): \ f(\mathrm{D}^\alpha(X - y) \oplus K) \subseteq \mathbb{A}, \ y \in \mathbb{R}^d\}. \tag{6.2}$$

By **D1**, $s_\alpha(X)$ can be given alternatively in terms of the acceptance set at level α as

$$s_\alpha(X) = \bigvee \{f(y): \ X - y \in \mathcal{A}_\alpha, \ y \in \mathbb{R}^d\}. \tag{6.3}$$

Theorem 6.5. *Assume that $\mathbb{F}$ is sup-generating. Then the mapping $s_\alpha(X)$ satisfies*

$$s_\alpha(X) = \bigwedge \mathbb{D}^\alpha(X), \tag{6.4}$$

*and so becomes a homogeneous risk measure associated with f. If the family of depth trimmed regions satisfies **D6**, then $s_\alpha(X)$ is a coherent risk measure such that $s_\alpha(X) \succeq s_\beta(X)$ for $\alpha \geq \beta$.*

Proof. The linearity of f and (6.2) imply that

$$\begin{aligned}
s_\alpha(X) &= \bigvee \{f(y): \ f(\mathrm{D}^\alpha(X) \oplus K) \subseteq \mathbb{A} \oplus f(y)\} \\
&= \bigvee \{f(y): \ c \in \mathbb{A} \oplus f(y) \ \text{ for all } \ c \in \mathbb{D}^\alpha(X)\} \\
&= \bigvee \{f(y): \ f(y) \preceq c \ \text{ for all } \ c \in \mathbb{D}^\alpha(X)\} \\
&= \bigvee \{a \in \mathbb{F}: \ a \preceq \bigwedge \mathbb{D}^\alpha(X)\},
\end{aligned}$$

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so that (6.4) follows from the sup-generating property (3.2).

If $X = x$ a.s., then $s_\alpha(X) = \bigwedge f(x + \mathrm{D}^\alpha(0) \oplus K) = f(x)$, since $\bigwedge f(\mathrm{D}^\alpha(0) \oplus K) = \mathbf{e}$ by **D5** and f is non-decreasing. By (2.1) and **D1** we deduce that $s_\alpha(X) \oplus f(y) = s_\alpha(X + y)$, so **R1** holds.

If $Y \leq_K X$ a.s., then $\mathrm{D}^\alpha(X) \subseteq \mathrm{D}^\alpha(Y) \oplus K$ by **D5**. Thus **R2** holds, since

$$s_\alpha(Y) = \bigwedge f(\mathrm{D}^\alpha(Y) \oplus K) \preceq \bigwedge f(\mathrm{D}^\alpha(X) \oplus K) = s_\alpha(X).$$

Property **R3** follows directly from **D2**, the fact that K is a cone and the homogeneity of f. If **D6** holds, then $\mathrm{D}^\alpha(X + Y) \oplus K \subseteq (\mathrm{D}^\alpha(X) \oplus K) \oplus (\mathrm{D}^\alpha(Y) \oplus K)$

$$s_\alpha(X + Y) = \bigwedge \mathrm{D}^\alpha(X + Y) \succeq \bigwedge \mathrm{D}^\alpha(X) \oplus \bigwedge \mathrm{D}^\alpha(Y) = s_\alpha(X) \oplus s_\alpha(X),$$

i.e. **R4** holds. Finally, the ordering of the risks with respect to α follows from **D3**. $\square$ $\square$

Now we describe a dual construction, based on rejection sets, of set-valued risk measures associated with depth-trimmed regions. The rejection set at level α associated with $\mathrm{D}^\alpha(\cdot)$ is given by

$$\mathcal{A}_\alpha^{\mathrm{r}} = \{X \in L_d^\infty : \mathbb{D}^\alpha(X) \cap \mathbb{A}^{\mathrm{r}} \neq \emptyset\} = \{X \in L_d^\infty : \mathrm{D}^\alpha(X) \cap (-K) \neq \emptyset\}.$$

Assuming that $\mathbb{F}$ is inf-generating, by (3.3) we have

$$s_\alpha^{\mathrm{r}}(X) = \bigwedge \varrho_{\mathcal{A}_\alpha^{\mathrm{r}}}(X),$$

where $s_\alpha^{\mathrm{r}}(X)$ is also given by (6.3) with $\mathcal{A}_\alpha$ replaced by $\mathcal{A}_\alpha^{\mathrm{r}}$. It is possible to reproduce Theorem 6.5 in this dual framework and obtain that

$$s_\alpha^{\mathrm{r}}(X) = \bigwedge \mathbb{D}^\alpha(X).$$

Further, s_α^{r} is a homogeneous risk measure which is also coherent if **D6** holds.

Example 6.6 (Set-valued risk measures from depth-trimmed regions). In the setting of Example 3.2 $f(x) = -x + K$, so that Theorem 6.5 implies that

$$s_\alpha(X) = \bigcap_{x \in \mathrm{D}^\alpha(X)} (-x + K) = \bigcap_{x \in \mathrm{D}^\alpha(X)} \{z \in \mathbb{R}^d : -x \leq_K z\}$$

$$= \{z \in \mathbb{R}^d : z + \mathrm{D}^\alpha(X) \subseteq K\} = \{z \in \mathbb{R}^d : \mathrm{D}^\alpha(X) \subseteq (-z + K)\}.$$

If K is a Riesz cone, then there exists the infimum of $\mathrm{D}^\alpha(X)$ with respect to the $\leq_K$-order (denoted as $\wedge_K \mathrm{D}^\alpha(X)$), so that

$$s_\alpha(X) = \bigcap_{x \in \mathrm{D}^\alpha(X)} \{z \in \mathbb{R}^d : -z \leq_K x\} = -\wedge_K \mathrm{D}^\alpha(X) + K. \tag{6.5}$$

Therefore, risk measures generated by depth-trimmed regions using the acceptance cone are not particularly interesting, since they are essentially vector-valued. In Example (9.2) it will

be shown that vector-valued risk measures are necessarily marginalised, i.e. they appear from the scheme of Example 2.6.

However, the rejection construction produces more interesting set-valued risk measures. Namely, in the setting of Example 3.2 with $f(x) = x + K^{\mathrm{r}}$, the corresponding risk measure $s_\alpha^{\mathrm{r}}(X)$ is the closure to the complement of $\mathrm{D}^\alpha(X) \oplus K$. The obtained risk measure takes values in the cone $\mathbb{C}_K^{\mathrm{r}}$.

7 Basic risk measures associated with depth-trimmed regions

Let us now specialise the constructions from Section 6 for $X = (X_1, X_2, \ldots, X_d) \in L_d^\infty$ and several basic definitions of depth-trimmed regions and set-valued risk measures with values either in $\mathbb{C}_K$ with a Riesz cone K. Recall that set-valued risk measures with values in $\mathbb{C}_K$ can be represented as $x + K$ for some $x \in \mathbb{R}^d$, i.e. are effectively vector-valued. Similar constructions are possible for $\mathbb{C}_K^{\mathrm{r}}$-valued risk measures. In this case the $\leq_K$-infimum of the complement to $\mathbb{C}_K^{\mathrm{r}}$-valued risk measures also yields a vector-valued risk measure, see Example 4.6.

Risk measures generated by monotone halfspace trimming. The monotone halfspace trimming induces a homogeneous risk measure, i.e. **R3** holds. This set-valued risk measure is given by $s_\alpha(X_1) = \big[\mathrm{V@R}_\alpha(X_1), +\infty\big)$ in the univariate case. In general,

$$s_\alpha(X_1, X_1, \ldots, X_1) = \big(\mathrm{V@R}_\alpha(X_1), \mathrm{V@R}_\alpha(X_1), \ldots, \mathrm{V@R}_\alpha(X_1)\big) + K\,,$$
$$s_\alpha(X) \supseteq \big(\mathrm{V@R}_\alpha(X_1), \mathrm{V@R}_\alpha(X_2), \ldots, \mathrm{V@R}_\alpha(X_d)\big) + K\,.$$

Risk measures generated by zonoid trimming. The zonoid trimming induces coherent risk measures. Then $s_\alpha(X_1) = \big[\mathrm{ES}_\alpha(X_1), +\infty\big)$ and in the multivariate setting

$$s_\alpha(X_1, X_1, \ldots, X_1) = \big(\mathrm{ES}_\alpha(X_1), \mathrm{ES}_\alpha(X_1), \ldots, \mathrm{ES}_\alpha(X_1)\big) + K\,,$$
$$s_\alpha(X) \supseteq \big(\mathrm{ES}_\alpha(X_1), \mathrm{ES}_\alpha(X_2), \ldots, \mathrm{ES}_\alpha(X_d)\big) + K\,,$$

where the latter inclusion turns into the equality if $K = \mathbb{R}_+^d$. If $K = A^{-1}\mathbb{R}_+^d$ for a matrix A, then

$$s_\alpha(X) = A^{-1}\big(\mathrm{ES}_\alpha((AX)_1), \mathrm{ES}_\alpha((AX)_2), \ldots, \mathrm{ES}_\alpha((AX)_d)\big) + K\,, \qquad (7.1)$$

where $(AX)_i$ stands for the i-th coordinate of AX. In particular, (7.1) implies that the marginalised expected shortfall (as in Example 2.6) of AX is coordinatewise smaller than A applied to the marginalised expected shortfall of X.

Risk measures generated by expected convex hull trimming. The expected convex hull trimming induces coherent risk measures. Then $s_{1/n}(X_1) = \big[\mathrm{EM}_{1/n}(X_1), +\infty\big)$ and

$$s_{1/n}(X_1, X_1, \dots, X_1) = \big(\mathrm{EM}_{1/n}(X_1), \mathrm{EM}_{1/n}(X_1), \dots, \mathrm{EM}_{1/n}(X_1)\big) + K,$$
$$s_{1/n}(X) \supseteq \big(\mathrm{EM}_{1/n}(X_1), \mathrm{EM}_{1/n}(X_2), \dots, \mathrm{EM}_{1/n}(X_d)\big) + K$$

with the equality if $K = \mathbb{R}^d_+$. If $K = A^{-1}\mathbb{R}^d_+$ for a matrix A, then (7.1) also holds for the expected minimum instead of the expected shortfall.

Note that in all three examples described above we have

$$s^{\mathrm{r}}_\alpha(X) \supseteq (\rho(X_1), \dots, \rho(X_d)) + K^{\mathrm{r}},$$

where ρ stands for $\mathrm{V@R}_\alpha$, ES_α or $\mathrm{EM}_{1/n}$.

Integral trimmed risk measures. The integral trimmed regions generate new multivariate risk measures. Consider the cone $\mathbb{C}_K$ from Example 2.4 and $f(x) = -x + K$. Let $\mathcal{F} = \{\mathfrak{f}(tx + z) : t > 0,\, z \in \mathbb{R}^d\}$, where $\mathfrak{f}$ is continuous and $\leq_K$-decreasing for a proper Riesz cone K. Since $\mathrm{D}^\alpha(X)$ is the union of

$$\mathrm{D}^1(Y) = \bigcap_{\substack{t>0 \\ z \in \mathbb{R}^d}} \left[\frac{1}{t}\Big(\mathfrak{f}^{-1}\big(\mathbf{E}\,\mathfrak{f}(tY + z)\big) - z \Big) \oplus K \right]$$

for $Y \in \mathcal{P}_\alpha(X)$, we obtain

$$s_\alpha(X) = \bigcap_{Y \in \mathcal{P}_\alpha(X)} s_1(Y),$$

so that $s_\alpha(X)$ appears from the worst conditioning construction applied to the risk measure s_1. Furthermore, (6.5) yields that $s_1(X) = x + K$, where

$$
\begin{aligned}
x = -\wedge_K \mathrm{D}^1(X) &= -\wedge_K \left(\bigcap_{\substack{t>0 \\ z \in \mathbb{R}^d}} \left[\frac{1}{t}\Big(\mathfrak{f}^{-1}\big(\mathbf{E}\,\mathfrak{f}(tY + z)\big) - z \Big) \oplus K \right] \right) \\
&= -\bigvee_{\substack{t>0 \\ z \in \mathbb{R}^d}} \wedge_K \left[\frac{1}{t}\Big(\mathfrak{f}^{-1}\big(\mathbf{E}\,\mathfrak{f}(tY + z)\big) - z \Big) \right] \\
&= \bigwedge_{\substack{t>0 \\ z \in \mathbb{R}^d}} \left(-\wedge_K \left[\frac{1}{t}\Big(\mathfrak{f}^{-1}\big(\mathbf{E}\,\mathfrak{f}(tY + z)\big) - z \Big) \right] \right).
\end{aligned}
\tag{7.2}
$$

This risk measure satisfies **R1**–**R3** and results from the homogenisation construction (4.1) and (4.2) applied to the set-valued risk measure generated by the integral trimmed regions whose generating family is $\mathcal{F} = \{\mathfrak{f}\}$,

$$\varrho(X) = -\wedge_K \mathfrak{f}^{-1}\big(\mathbf{E}\,\mathfrak{f}(X)\big) + K. \tag{7.3}$$

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Notice that this homogenisation preserves **R2**, but not necessarily **R4**. The idea of constructing scalar risk measures using real-valued functions of vector portfolios appears also in [4]. Alternatively, it is possible to take infimum in over $t > 0$ or over $z \in \mathbb{R}^d$ only, which results in a risk measure that satisfies **R3** or **R1** respectively.

Example 7.1. The function $\mathfrak{f}(t) = e^{-t/\gamma}$ yields the risk measure $\varrho(X) = \gamma \log(\mathbf{E} e^{-X/\gamma})$ by (7.3) in $\mathbb{C} = \mathbb{R}$ with the reversed order and $f(x) = -x$. The properties **R1** and **R2** evidently holds, while (2.2) follows from the Hölder inequality, i.e. ϱ is a *convex* risk measure, which does not satisfy **R3**. Since **R1** already holds, there is no need to take infimum over $z \in \mathbb{R}^d$ in (7.2). The corresponding convex risk measure is called the *entropic risk measure* with γ being the risk tolerance coefficient.

If we attempt to produce a homogeneous (and thereupon coherent) risk measure from ϱ, we need to apply (4.1), which in view of the reversed order on the real line turns into

$$\varrho_{\mathrm{h}}(X) = \sup_{t>0} \, t^{-1} \, \varrho(tX) = \sup_{t>0} \, t^{-1} \log(\mathbf{E} e^{-tX}) = \sup_{t>0} \, \log((\mathbf{E} Y^t)^{1/t})$$

for $Y = e^{-X}$. Since $(\mathbf{E} Y^t)^{1/t}$ is an increasing function of $t > 0$, we have

$$\varrho_{\mathrm{h}}(X) = \lim_{t \to \infty} t^{-1} \log(\mathbf{E} e^{-tX}) \,.$$

It is easy to see that the limit equals $(-\operatorname{essinf} X)$, so a coherent variant of ϱ is not particularly interesting.

8 Depth-trimmed regions generated by risk measures

Consider a family of homogeneous risk measures ϱ_α for $\alpha \in (0, 1]$ such that

$$\varrho_\alpha \succeq \varrho_\beta \,, \quad \alpha \geq \beta \,, \tag{8.1}$$

which are associated with function f according to **R1**. For instance, such family of risk measures can be produced using the worst conditioning construction from Section 4.4.

Definition 8.1. *The* depth-trimmed regions generated by the family of risk measures *are defined as*
$$\mathrm{D}^\alpha(X) = \{y \in \mathbb{R}^d : \ \varrho_\alpha(X - y) \preceq \mathbf{e}\} \,.$$

By **R1**, the depth-trimmed regions generated by a family of risk measures are alternatively given by
$$\mathrm{D}^\alpha(X) = \{y \in \mathbb{R}^d : \ \varrho_\alpha(X) \preceq f(y)\} \,. \tag{8.2}$$

Theorem 8.2. *The depth-trimmed regions generated by a family of risk measures satisfy*

   *(i) properties **D1**, **D2**, **D3** and **D5**;*

   (ii) are convex if the risk measure is convex;

(iii) are closed if f is upper semicontinuous, i.e. $\{x \in \mathbb{R}^d : a \preceq f(x)\}$ *is closed in* $\mathbb{R}^d$ *for every* $a \in \mathbb{C}$.

Proof. (i) Properties **D1** and **D2** trivially hold by **R1** and **R3** respectively. The nesting property **D3** of depth-trimmed regions is a consequence of (8.1). We will show that **D5** follows from **R2**. If $Y \leq_K X$ a.s., then **R2** yields that $\varrho_\alpha(Y) \preceq \varrho_\alpha(X)$. Then

$$\{y \in \mathbb{R}^d : \varrho_\alpha(Y) \preceq f(y)\} \supseteq \{y \in \mathbb{R}^d : \varrho_\alpha(X) \preceq f(y)\}$$

and by (8.2) we have $\mathrm{D}^\alpha(Y) \supseteq \mathrm{D}^\alpha(X)$ and finally $\mathrm{D}^\alpha(X) \subseteq \mathrm{D}^\alpha(Y) \oplus K$, since $0 \in K$.

(ii) Given $y, z \in \mathrm{D}^\alpha(X)$ and $t \in [0, 1]$,

$$\varrho_\alpha(X) \preceq t \odot f(y) \oplus (1 - t) \odot f(z) = f(ty + (1 - t)z)$$

and finally $ty + (1 - t)z \in \mathrm{D}^\alpha(X)$.

(iii) If f is upper semicontinuous, the set $\{y \in \mathbb{R}^d : \varrho_\alpha(X) \preceq f(y)\}$ is closed. $\qquad\square\qquad\square$

Under mild conditions, it is possible to recover a risk measure from the depth-trimmed regions generated by it. If $\mathbb{F}$ is sup-generating and inf-generating, the original risk measure is the risk measure induced by the family of depth-trimmed regions that it generated. Theorem 6.5 and equation (8.2) yield that

$$s_\alpha(X) = \bigwedge \mathbb{D}^\alpha(X) = \bigwedge \{f(y) : \varrho_\alpha(X) \preceq f(y)\} = \varrho_\alpha(X) \,.$$

Notice that if the construction based on rejection sets is used, see (3.3), the first equality in the left holds when $\mathbb{F}$ is inf-generating, so the sup-generating assumption on $\mathbb{F}$ can be dropped and we still have

$$s_\alpha^{\mathrm{r}}(X) = \varrho_\alpha(X) \,.$$

Example 8.3 (Expected convex hull trimming revisited). The expected minimum can be formulated as a *spectral risk measure*, see [1], as

$$\mathrm{EM}_{1/n}(X) = -\int_0^1 n(1 - t)^{n-1} F_X^{-1}(t) dt \,, \quad n \geq 1 \,. \tag{8.3}$$

For any $\alpha \in (0, 1]$, define $\mathrm{EM}_\alpha(X)$ substituting n by α^{-1} in (8.3). The risk measures EM_α generates a family of depth-trimmed regions for $X \in L_1^\infty$ with a continuous parameter $\alpha \in (0, 1]$. Applying Definition 8.1, we obtain $\mathrm{D}^\alpha(X) = [-\mathrm{EM}_\alpha(X), +\infty)$.

In contrast to risk measures, depth-trimmed regions treat all directions in the same way, so that the regions D^α must be slightly modified so that they yield the expected convex hull trimmed regions for $\alpha = 1/n$. Define

$$\begin{aligned}
\mathrm{CD}^\alpha(X) &= \mathrm{D}^\alpha(X) \cap \left(-\mathrm{D}^\alpha(-X)\right) \\
&= [-\mathrm{EM}_\alpha(X), \mathrm{EM}_\alpha(-X)] \\
&= \left[\alpha^{-1} \int_0^1 (1 - t)^{\alpha^{-1}-1} F_X^{-1}(t) dt, \, \alpha^{-1} \int_0^1 t^{\alpha^{-1}-1} F_X^{-1}(t) dt\right].
\end{aligned}$$

In this formulation, we can assume that the parameter α takes any value in $(0, 1]$ and thus, we obtain an extension of the univariate expected convex hull trimmed regions.

9 Duality results

The dual space to L_d^∞ is the family of finitely additive bounded *vector* measures $\mu = (\mu_1, \ldots, \mu_d)$ on the underlying probability space $(\Omega, \mathfrak{F})$, which act on $X \in L_d^\infty$ as $\mathbf{E}_\mu(X) = \sum_{i=1}^d \int X_i d\mu_i$. The polar set to the cone of acceptable portfolios to a coherent risk measure ϱ can be written as

$$\mathcal{A}^* = \bigcap_{X \in \mathcal{A}} \{\mu : \mathbf{E}_\mu(X) \geq 0\}.$$

As in [16], we can apply the bipolar theorem to show that

$$\mathcal{A} = \bigcap_{\mu \in \mathcal{A}^*} \{X : \mathbf{E}_\mu(X) \geq 0\}. \tag{9.1}$$

For each $u \in K$ and measurable $\Omega' \subset \Omega$, the random element $u\mathbf{1}_{\Omega'}$ belongs to $\mathcal{A}$. Therefore, every $\mu \in \mathcal{A}^*$ satisfies $\sum \mu_i(\Omega')u_i \geq 0$ for every $u \in K$. Thus, the values of any $\mu \in \mathcal{A}^*$ belong to the positive dual cone to K.

Assume that $\mathbb{F}$ is sup-generating. Proposition 3.1 implies that

$$\varrho(X) = \bigvee \{f(y) : X - y \in \mathcal{A}\}.$$

It follows from (9.1) that

$$\varrho(X) = \bigvee \{f(y) : y \in \mathbb{R}^d, \ \mathbf{E}_\mu(X) \geq \langle \mu, y \rangle \text{ for all } \mu \in \mathcal{A}^*\}$$

$$= \bigvee f\left(\bigcap_{\mu \in \mathcal{A}^*} \{y \in \mathbb{R}^d : \langle \mu, y \rangle \leq \mathbf{E}_\mu(X)\} \right),$$

where $\langle \mu, x \rangle = \sum_{i=1}^d \mu_i(\Omega)x_i$.

For instance, a set-valued risk measure with values in the cone of convex closed sets in $\mathbb{R}^d$ with the inclusion order (so that $f(-x) = x + K$) can be represented as

$$\varrho(X) = \bigcap_{\mu \in \mathcal{A}^*} \{x \in \mathbb{R}^d : \langle \mu, x \rangle \geq \mathbf{E}_\mu(-X)\},$$

where $\mathcal{A}^*$ is a set of finitely additive bounded vector measures with values in K. Note that there is no need to add K to the right-hand side, since $\langle \mu, x + z \rangle \geq \langle \mu, x \rangle$ in case μ takes values from the positive dual to K. By applying to this set-valued risk measure ϱ the worst conditioning construction, we obtain

$$\varrho_\alpha(X) = \bigwedge_{Y \in \mathcal{P}_\alpha(X)} \bigcap_{\mu \in \mathcal{A}^*} \{x \in \mathbb{R}^d : \langle \mu, x \rangle \geq \mathbf{E}_\mu(-Y)\}$$

$$= \bigcap_{\mu \in \mathcal{A}^*} \{x \in \mathbb{R}^d : \langle \mu, x \rangle \geq (-\mathbf{E}_\mu)_\alpha(X)\},$$

where
$$(-\mathbf{E}_\mu)_\alpha(X) = (-\mathbf{E}_{\mu_1})_\alpha X_1 + \cdots + (-\mathbf{E}_{\mu_d})_\alpha X_d \,.$$
Thus ϱ_α also admits the dual representation, where instead of the expectation $\mathbf{E}_\mu(-X)$ we take the expected shortfall of X with respect to the measure μ. Definition 8.1 yields then a dual representation for the family of depth-trimmed regions.

If the risk measure satisfies the Fatou property, then all measures from $\mathcal{A}^*$ can be chosen to be σ-additive. Recall that the Fatou property means that the risk measure is lower semicontinuous in probability, i.e. the lower limit (which for set-valued risk measures is understood in the Painlevé-Kuratowski sense [21, Def. B.5]) of $\varrho(X_k)$ is not greater than $\varrho(X)$ if X_k converges in probability to X.

Example 9.1 (Vector-valued coherent risk measures). Let ϱ be a coherent risk measure with values in $\mathbb{C} = \mathbb{R}^d$ with the reversed $\leq_K$-order with $K = \mathbb{R}^d_+$ and $f(x) = -x$. Then
$$\varrho(X) = \bigvee \bigcap_{\mu \in \mathcal{A}^*} Y(\mu) \,,$$
where all $\mu \in \mathcal{A}^*$ have non-negative coordinates and
$$Y(\mu) = \{-y : \ \langle \mu, y \rangle \leq \mathbf{E}_\mu(X), \ y \in \mathbb{R}^d\} = \{y : \ \langle \mu, y \rangle \geq -\mathbf{E}_\mu(X), \ y \in \mathbb{R}^d\} \,.$$
For instance, the first coordinate of $\varrho(X)$ is obtained as the infimum of the projection of $\cap_{\mu \in \mathcal{A}^*} Y(\mu)$ onto the first coordinate. If $\mu_1(\Omega) > 0$ and $\mu_2(\Omega) = \cdots = \mu_d(\Omega) = 0$, then $Y(\mu) = [y_1, \infty) \times \mathbb{R} \cdots \times \mathbb{R}$ for some y_1. If x^* is the essential lower bound of X with respect to $\leq_K$-order, then $(y_1, -x_2^*, \ldots, -x_d^*)$ belongs to $Y(\mu)$ for all $\mu \in \mathcal{A}^*$. Thus, the first coordinate of $\varrho(X)$ is given by the infimum $-\mu_1(\Omega)^{-1}\mathbf{E}_{\mu_1}X_1$ over all measures $(\mu_1, 0, \ldots, 0)$ that belong to $\mathcal{A}^*$. Thus,
$$\varrho(X) = \left(-\inf_{\mu_1 \in \mathcal{A}_1^*} \mathbf{E}_{\mu_1}(X_1), \ldots, -\inf_{\mu_d \in \mathcal{A}_d^*} \mathbf{E}_{\mu_d}(X_d) \right), \tag{9.2}$$
where $\mathcal{A}_i^*$ is the family of normalised measures μ_i such that $(0, \ldots, 0, \mu_i, 0, \ldots, 0) \in \mathcal{A}^*$, $i = 1, \ldots, d$. The individual infima in (9.2) are risk measures themselves. Therefore, $\varrho(X)$ can be represented as the vector composed of coherent risk measures of the marginals of X.

A similar argument is applicable for the risk measure $A^{-1}\varrho(AX)$ if K is a general Riesz cone with $K = A^{-1}\mathbb{R}^d_+$. Then $A\varrho(X)$ can be represented as the vector composed of risk measures calculated for the coordinates of AX, cf. (7.1).

10   Conclusions

It is likely that other results from the morphological theory of lattices [13] have applications in the framework of risk measures. In particular, it would be interesting to find a financial interpretation for dilation mappings that commute with supremum, erosions that commute with infimum, and pairs of erosions and dilations that are called adjunctions.

It is possible to consider a variant of **R2** where $Y \leq X$ is understood with respect to any other chosen order on L_d^∞. The consistency issues for scalar risk measures for vector portfolios are investigated in [4] and in [3] for the one-dimensional case.

Acknowledgements

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